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Perturbation theory in the radial quantization approach and the expectation values of exponential fields in the sine–Gordon model

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Abstract. We have developed a perturbation theory, based on the radial quantization of the massive Thirring model (MTM). It is remarkable that the apparent difficulty in radial quantization of massive theories, namely, the explicit 'time' dependence of the Hamiltonian, may be successfully overcome. In this framework, in first order of the coupling constant of MTM, we calculate the vacuum–vacuum amplitude with arbitrary twisted boundary conditions imposed on the Fermi fields. In terms of sine–Gordon theory these amplitudes are nothing other than the expectation values of exponential fields $\langle \exp \mathrm{i}a\varphi(0)\rangle$. The result we have obtained coincides with the analogous calculations recently carried out in a dual, angular quantization approach by one of the authors and completely agrees with the exact formula conjectured by Lukyanov and Zamolodchikov.

1. Introduction

The main subject of investigation in this paper is the sine–Gordon model, which is one of the most studied examples of two-dimensional (2D) integrable quantum field theory (IQFT). Its action is given by

$$\mathcal{S}_{SG} = \int \mathrm{d}^2x \left\{ \frac{1}{16\pi} \partial_\nu \varphi \partial^\nu \varphi + 2\mu \cos \beta\varphi \right\} \tag{1.1}$$

where φ is a real Bose field. The spectrum of the model includes the soliton, anti soliton and some of their bound states, named breathers. The number of breathers depends on the coupling constant β. The S-matrix, describing scattering of these particles is known exactly [1]. Due to the famous work by Coleman [2] the sine–Gordon model is equivalent to a fermionic model with a four-fermion interaction, namely, to the massive Thirring model (MTM), which has the following action:

$$\mathcal{S}_{MTM} = \int \mathrm{d}^2x \left\{ \mathrm{i}\overline{\psi}\gamma^\nu \partial_\nu \psi - M\,\overline{\psi}\psi - \tfrac{1}{2} g \left(\overline{\psi}\gamma^\nu\psi\right)\left(\overline{\psi}\gamma_\nu\psi\right)\right\} \tag{1.2}$$

where $\overline{\psi}$ and ψ are two-component Dirac spinors. The fundamental (anti-)fermions of the MTM should be identified with the (anti-)solitons of the sine–Gordon model. The parameters and currents of MTM and sine–Gordon theories are related as [2]

$$\frac{g}{\pi} = \frac{1}{2\beta^2} - 1 \qquad J^\nu \equiv \overline{\psi}\gamma^\nu\psi = -\frac{\beta}{2\pi}\epsilon^{\nu\mu}\partial_\mu\varphi. \tag{1.3}$$

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More recently Zamolodchikov obtained an exact relation between the soliton mass M and the parameter μ in the action (1.1) [3]

$$\mu = \frac{\Gamma\left(\beta^2\right)}{\pi\Gamma\left(1-\beta^2\right)}\left[\frac{M\sqrt{\pi}\,\Gamma(\frac{1}{2}(1+\xi))}{2\Gamma(\xi/2)}\right]^{2-2\beta^2} \tag{1.4}$$

where

$$\xi = \frac{\beta^2}{1-\beta^2}. \tag{1.5}$$

In this paper we consider the vacuum expectation values (VEV) of exponential fields in the sine–Gordon model

$$G_a = \langle \exp \mathrm{i}a\varphi(0)\rangle \tag{1.6}$$

where the exponential fields are normalized by the condition

$$\left\langle \mathrm{e}^{\mathrm{i}a\varphi(x)}\mathrm{e}^{-\mathrm{i}a\varphi(y)}\right\rangle_{SG} \to |x-y|^{-4a^2} \qquad \text{as} \quad |x-y|\to 0 \tag{1.7}$$

which emphasizes that the ultraviolet limit of this theory is governed by the $c = 1$ free-boson conformal field theory.

For two special values of the sine–Gordon coupling constant, namely, for $\beta \to 0$ (the semiclassical limit) and $\beta^2 = \frac{1}{2}$ (the free-fermion case), this function admits a direct calculation. The authors of [4] have used these special cases to guess the following expression for the expectation values (1.6) for generic $\beta^2 < 1$ and $|\operatorname{Re}(a)| < 1/(2\beta)$

$$G_a = \left(\frac{m\Gamma(\frac{1}{2}(1+\xi))\Gamma\left(\frac{1}{2}(2-\xi)\right)}{4\sqrt{\pi}}\right)^{2a^2} \times \exp\left\{\int_0^\infty \frac{\mathrm{d}t}{t}\left[\frac{\sinh^2(2a\beta t)}{2\sinh(\beta^2 t)\sinh t\cosh\left(\left(1-\beta^2\right)t\right)} - 2a^2\mathrm{e}^{-2t}\right]\right\}. \tag{1.8}$$

In order to support the formula (1.8), some extra arguments, based on the reflection relations with Liouville reflection amplitude [5] have been presented in the subsequent papers [6, 7], but a rigorous proof is lacking up to now. The article [8] provides more evidence supporting the Lukyanov–Zamolodchikov formula (1.8), where using perturbation theory in the angular quantization approach [9], formula (1.8) has been checked in first order of the MTM coupling constant g.

Here we apply radial quantization to the same problem. The Hamiltonian of massive theories in the radial quantization approach has an explicit time dependence [10]. It appeared that this apparent difficulty can be successfully overcome. Note, that to carry out the same calculation using the ordinary Feynman diagram technique, one should sum up an infinite number of two-loop diagrams. We hope, that such calculations should substantially increase the confidence in the reflection relations method as a whole, which appears to be a very powerful tool for the investigation of 2D conformal field theory (CFT) and IQFT [11, 12].

This paper is organized as follows. In section 2 we present the radial quantization of MTM. In section 3 we calculate the VEV (1.6) at the free-fermion point $g = 0$. The calculation of VEV in the first order of perturbation theory is presented in section 4. Here special attention has been paid to the regularization procedure of the product of local fields at the coinciding points, which has some new features in comparison with the ordinary quantization in Cartesian coordinates. It appears that the Hankel transform is a useful tool to carry out the calculations of section 4. The relevant mathematical details are presented in an appendix.

2. Radial quantization of the massive Thirring model

In two-dimensional space Dirac matrices γ^0 and γ^1 have the following convenient representation:

$$\gamma^0 = \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \qquad \gamma^1 = -\mathrm{i}\sigma_1 = -\mathrm{i}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2.1}$$

We denote the components of Dirac spinors as

$$\psi \equiv \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \qquad \overline{\psi} \equiv \psi^\dagger \gamma^0. \tag{2.2}$$

In this notation the action (1.7) in Euclidean space takes the form

$$\mathcal{A}_{MTM} = \int \mathrm{d}^2 z \left[\psi_R^\dagger \partial \psi_R + \psi_L^\dagger \overline{\partial} \psi_L - \tfrac{1}{2}\mathrm{i}M\left(\psi_L^\dagger \psi_R - \psi_R^\dagger \psi_L\right) + g\psi_L^\dagger \psi_L \psi_R^\dagger \psi_R\right] \tag{2.3}$$

where $z = x^2 + \mathrm{i}x^1$, $\overline{z} = x^2 - \mathrm{i}x^1$ are the complex coordinates on the Euclidean plane, $\partial \equiv \partial/\partial z$, $\overline{\partial} \equiv \partial/\partial\overline{z}$ and $\mathrm{d}^2 z \equiv 2\,\mathrm{d}x^1\,\mathrm{d}x^2$ is the volume element.

As we are intending to evaluate the VEV of the field $\langle \mathrm{e}^{\mathrm{i}a\varphi(0)}\rangle$, which obviously is invariant with respect to the rotations around the centre of coordinates, it is convenient to use the conformal polar coordinates η, θ defined by

$$z = \mathrm{e}^{\eta + \mathrm{i}\theta} \qquad \overline{z} = \mathrm{e}^{\eta - \mathrm{i}\theta}. \tag{2.4}$$

In what follows, we will interpret η and θ as Euclidean time and space, respectively.

Since the conformal weights of the Fermi fields ψ_L and ψ_R are $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$, respectively, their transformations under the conformal map (2.4) are given by

$$\psi_L(z, \overline{z}) \rightarrow \left(\frac{\partial\xi}{\partial z}\right)^{1/2} \psi_L(\eta, \theta) \qquad \psi_R(z, \overline{z}) \rightarrow \left(\frac{\partial\overline{\xi}}{\partial\overline{z}}\right)^{1/2} \psi_R(\eta, \theta) \tag{2.5}$$

where $\xi = \eta + \mathrm{i}\theta$, $\overline{\xi} = \eta - \mathrm{i}\theta$. The same transformation laws hold for the fields $\psi_{L,R}^\dagger$.

In new coordinates (η, θ) the action (2.3) can be rewritten as

$$\begin{aligned}\mathcal{A}_{MTM} = \int_0^{2\pi} \mathrm{d}\theta \int_{-\infty}^{\infty} \mathrm{d}\eta \, &\left[\mathrm{i}\psi_L^\dagger\left(\partial_\theta - \mathrm{i}\partial_\eta\right)\psi_L - \mathrm{i}\psi_R^\dagger(\partial_\theta + \mathrm{i}\partial_\eta)\psi_R\right. \\ &\left. -\mathrm{i}M\mathrm{e}^\eta\left(\psi_L^\dagger\psi_R - \psi_R^\dagger\psi_L\right) + 2g\psi_L^\dagger\psi_L\psi_R^\dagger\psi_R\right].\end{aligned} \tag{2.6}$$

Treating the radial coordinate η as a Euclidean time, one immediately obtains the Hamiltonian

$$H = \int_0^{2\pi} \mathrm{d}\theta\left[\psi_L^\dagger \mathrm{i}\partial_\theta\psi_L - \psi_R^\dagger \mathrm{i}\partial_\theta\psi_R - \mathrm{i}M\mathrm{e}^\eta\left(\psi_L^\dagger\psi_R - \psi_R^\dagger\psi_L\right) + 2g\psi_L^\dagger\psi_L\psi_R^\dagger\psi_R\right]. \tag{2.7}$$

The usual canonical quantization scheme ensures the following standard equal-time anticommutation relations:

$$\left\{\psi_L(\theta), \psi_L^\dagger\left(\theta'\right)\right\} = \delta\left(\theta - \theta'\right) \qquad \left\{\psi_R(\theta), \psi_R^\dagger\left(\theta'\right)\right\} = \delta\left(\theta - \theta'\right). \tag{2.8}$$

As usual, in order to develop perturbation theory one first has to solve the problem with a quadratic Hamiltonian, ignoring the last quartic term in (2.7). Due to the explicit time dependence of the Hamiltonian it is easier to handle the problem in the Schrödinger picture rather than in the more conventional QFT Heisenberg picture. Thus our field operators $\psi_{L,R}$ do not depend on the 'time' η and, instead, the state vectors evolve according to the Schrödinger equation. Let us define the creation, annihilation operators $c_k^\dagger$, $d_k^\dagger$, c_k, d_k through the Fourier mode decompositions

$$\begin{aligned}
\psi_L(\theta) &= \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathcal{N}-1/2} \left(c_k \mathrm{e}^{-\mathrm{i}k\theta} + d_k^\dagger \mathrm{e}^{\mathrm{i}k\theta}\right) \\
\psi_R(\theta) &= \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathcal{N}-1/2} \left(c_{-k} \mathrm{e}^{\mathrm{i}k\theta} + d_{-k}^\dagger \mathrm{e}^{-\mathrm{i}k\theta}\right) \\
\psi_L^\dagger(\theta) &= \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathcal{N}-1/2} \left(d_k \mathrm{e}^{-\mathrm{i}k\theta} + c_k^\dagger \mathrm{e}^{\mathrm{i}k\theta}\right) \\
\psi_R^\dagger(\theta) &= \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathcal{N}-1/2} \left(d_{-k} \mathrm{e}^{\mathrm{i}k\theta} + c_{-k}^\dagger \mathrm{e}^{-\mathrm{i}k\theta}\right)
\end{aligned} \tag{2.9}$$

where all sums are taken over all positive half-integers ($\mathcal{N}$ is the set of positive integers).

As a consequence of equations (2.8) and (2.9), one can easily obtain the following anticommutation relations for the operators $c_k, d_k, c_k^\dagger, d_k^\dagger$:

$$\begin{aligned}
&\{c_k, c_l\} = \left\{c_k^\dagger, c_l^\dagger\right\} = \{d_k, d_l\} = \left\{d_k^\dagger, d_l^\dagger\right\} = 0 \\
&\left\{c_k, c_l^\dagger\right\} = \delta_{k,l} \qquad \left\{d_k, d_l^\dagger\right\} = \delta_{k,l} \qquad k, l \in \mathcal{Z} - \tfrac{1}{2}
\end{aligned} \tag{2.10}$$

where $\mathcal{Z}$ is the set of integers. As usual, the Fock space (let us denote it by $\mathcal{H}$) has the following basis vectors:

$$\prod_{k\in\mathcal{Z}-1/2} \left(c_k^\dagger\right)^{n_k} \left(d_k^\dagger\right)^{\tilde{n}_k} |0\rangle \tag{2.11}$$

where $n_k \in \{0, 1\}$ ($\tilde{n}_k \in \{0, 1\}$) are the occupation numbers of 'particles' created by the operators $c_k^\dagger$ ($d_k^\dagger$) out of the bare vacuum $|0\rangle$, which by definition satisfies the conditions

$$c_k|0\rangle = d_k|0\rangle = 0 \qquad k \in \mathcal{Z} - \tfrac{1}{2}. \tag{2.12}$$

Let us decompose the Hamiltonian (2.7) into the sum of a quadratic part H_0 and an interaction term H_{int}. In terms of the creation, annihilation operators the quadratic part H_0 can be rewritten as

$$\begin{aligned}
H_0 = \sum_{k\in\mathcal{N}-1/2} &\left[k\left(c_k^\dagger c_k - d_k d_k^\dagger + c_{-k}^\dagger c_{-k} - d_{-k} d_{-k}^\dagger\right)\right. \\
&\left. -\mathrm{i}M\mathrm{e}^\eta\left(c_k^\dagger d_{-k}^\dagger - d_{-k} c_k + d_k c_{-k} - c_{-k}^\dagger d_k^\dagger\right)\right].
\end{aligned} \tag{2.13}$$

The evolution of an arbitrary state $|s\rangle$ along Euclidean time η caused by the Hamiltonian H_0 is given by the Schrödinger equation

$$-r\frac{\partial}{\partial r}|s, r\rangle = H_0|s, r\rangle. \tag{2.14}$$

Here and henceforth we prefer to use $r \equiv M\mathrm{e}^\eta$ rather than η. To find the general solution to the Schrödinger equation (2.14) let us denote the Hamiltonian H_0 in the factorized form

$$H_0 = \sum_{k\in\mathcal{N}-1/2} \left(H_k^{(1)} + H_k^{(2)}\right) \tag{2.15}$$

where the operator $H_k^{(1)}$ ($H_k^{(2)}$) includes only $c_k, d_{-k}, c_k^\dagger, d_{-k}^\dagger$ ($d_k, c_{-k}, d_k^\dagger, c_{-k}^\dagger$). This makes it convenient to represent the Fock space $\mathcal{H}$ as an infinite tensor product

$$\mathcal{H} = \otimes_{k\in\mathcal{N}-1/2}\big(\mathcal{H}_k^{(1)} \otimes \mathcal{H}_k^{(2)}\big) \tag{2.16}$$

where $\mathcal{H}_k^{(1)}$ and $\mathcal{H}_k^{(2)}$ are four-dimensional vector spaces with base vectors

$$\begin{array}{lll} |0_k^{(1)}\rangle & c_k^\dagger d_{-k}^\dagger|0_k^{(1)}\rangle & \text{(even sector)} \\ c_k^\dagger|0_k^{(1)}\rangle & d_{-k}^\dagger|0_k^{(1)}\rangle & \text{(odd sector)} \end{array} \tag{2.17}$$

and

$$\begin{array}{lll} |0_k^{(2)}\rangle & d_k^\dagger c_{-k}^\dagger|0_k^{(2)}\rangle & \text{(even sector)} \\ d_k^\dagger|0_k^{(2)}\rangle & c_{-k}^\dagger|0_k^{(2)}\rangle & \text{(odd sector)} \end{array} \tag{2.18}$$

respectively. The vectors $|0_k^{(1)}\rangle$ and $|0_k^{(2)}\rangle$ are defined by the conditions

$$\begin{aligned} c_k|0_k^{(1)}\rangle &= d_{-k}|0_k^{(1)}\rangle = 0 \\ d_k|0_k^{(2)}\rangle &= c_{-k}|0_k^{(2)}\rangle = 0 \end{aligned} \tag{2.19}$$

where $k \in \mathcal{N} - \frac{1}{2}$. Note that the bare vacuum $|0\rangle$, introduced earlier (see (2.12)) can be represented as

$$|0\rangle = \otimes_{k\in\mathcal{N}-1/2}\Big[|0_k^{(1)}\rangle \otimes |0_k^{(2)}\rangle\Big]. \tag{2.20}$$

The operator $H_k^{(1)}$ ($H_k^{(2)}$) non-trivially acts only on the factor $\mathcal{H}_k^{(1)}$ ($\mathcal{H}_k^{(2)}$) of the full Fock space $\mathcal{H}$ (2.16), hence we have reduced the initial QFT problem of infinitely many degrees of freedom to the simple quantum mechanical one, with four-dimensional Hilbert space. A further simplification provides the observation that the reduced Hamiltonians $H_k^{(1)}$, $H_k^{(2)}$ do not mix even and odd sectors (see (2.17) and (2.18)). The resulting Schrödinger equations in this reduced spaces take the form

$$\begin{aligned} -r\frac{\partial}{\partial r}\Big[\alpha_k(r) + \beta_k(r)c_k^\dagger d_{-k}^\dagger\Big]\big|0_k^{(1)}\big\rangle &= H_k^{(1)}\Big[\alpha_k(r) + \beta_k(r)c_k^\dagger d_{-k}^\dagger\Big]\big|0_k^{(1)}\big\rangle \\ &= \Big[-k\alpha_k(r) + \mathrm{i}r\beta_k(r) + (k\beta_k(r) - \mathrm{i}r\alpha_k(r))c_k^\dagger d_{-k}^\dagger\Big]\big|0_k^{(1)}\big\rangle \end{aligned} \tag{2.21}$$

and

$$-r\frac{\partial}{\partial r}\Big[\gamma_k(r)c_k^\dagger + \delta_k(r)d_{-k}^\dagger\Big]\big|0_k^{(1)}\big\rangle = H_k^{(1)}\Big[\gamma_k(r)c_k^\dagger + \delta_k(r)d_{-k}^\dagger\Big]\big|0_k^{(1)}\big\rangle = 0. \tag{2.22}$$

Evidently, to obtain the equations for another sector with the Hamiltonian $H_k^{(2)}$, one simply has to change the upper indices (1) into (2) and make the substitutions $c_k \to d_k$ and $d_{-k} \to c_{-k}$. Thus, in both cases the unknown functions $\alpha_k(r)$, $\beta_k(r)$, $\gamma_k(r)$, $\delta_k(r)$ obey the differential equations

$$\begin{aligned} &\left(\frac{\partial}{\partial r} - \frac{k}{r}\right)\alpha_k(r) = -\mathrm{i}\beta_k(r) \\ &\left(\frac{\partial}{\partial r} + \frac{k}{r}\right)\beta_k(r) = \mathrm{i}\alpha_k(r) \\ &\frac{\partial}{\partial r}\gamma_k(r) = \frac{\partial}{\partial r}\delta_k(r) = 0. \end{aligned} \tag{2.23}$$

Therefore, γ_k and δ_k do not depend on r, while α_k and β_k can be expressed via modified Bessel functions I_ν, K_ν as follows:

$$\begin{aligned}\alpha_k(r) &= r^{1/2}\left(a_k I_{k-1/2}(r) + b_k K_{k-1/2}(r)\right)\\ \beta_k(r) &= \mathrm{i}r^{1/2}\left(a_k I_{k+1/2}(r) - b_k K_{k+1/2}(r)\right).\end{aligned} \tag{2.24}$$

One should fix the constants a_k, b_k, γ_k and δ_k by imposing initial conditions at the arbitrary 'time' r_0. For further application let us write down explicit expressions with specified constants for two basic cases.

(a) When the initial state coincides with $|0_k^{(1)}\rangle$ or $|0_k^{(2)}\rangle$:

$$\begin{aligned}\alpha_k(r) &= \sqrt{rr_0}\left(K_{k+1/2}\left(r_0\right) I_{k-1/2}(r) + I_{k+1/2}\left(r_0\right) K_{k-1/2}(r)\right)\\ \beta_k(r) &= \mathrm{i}\sqrt{rr_0}\left(K_{k+1/2}\left(r_0\right) I_{k+1/2}(r) - I_{k+1/2}\left(r_0\right) K_{k+1/2}(r)\right).\end{aligned} \tag{2.25}$$

(b) When the initial state coincides with $\mathrm{i}c_k^{(\dagger)} d_{-k}^{(\dagger)}|0_k^{(1)}\rangle$ or $\mathrm{i}d_k^{(\dagger)} c_{-k}^{(\dagger)}|0_k^{(2)}\rangle$

$$\begin{aligned}\alpha_k(r) &= \sqrt{rr_0}\left(K_{k-1/2}\left(r_0\right) I_{k-1/2}(r) - I_{k-1/2}\left(r_0\right) K_{k-1/2}(r)\right)\\ \beta_k(r) &= \mathrm{i}\sqrt{rr_0}\left(K_{k-1/2}\left(r_0\right) I_{k+1/2}(r) + I_{k-1/2}\left(r_0\right) K_{k+1/2}(r)\right).\end{aligned} \tag{2.26}$$

To prove the formulae (2.25) and (2.26) we have used the following Wronskian identity for the modified Bessel functions [13]:

$$I_{k-1/2}(r)K_{k+1/2}(r) + I_{k+1/2}(r)K_{k-1/2}(r) = \frac{1}{r}. \tag{2.27}$$

It is interesting to note that due to the explicit time dependence of the Hamiltonian, the system which is initially in the ground state of that particular moment will after finite evolution time find itself in an excited state. Nevertheless, long time evolution of any state with non-vanishing overlap with the ground state of the initial time, eventually approaches to the ground state of the infinite future

$$|\infty\rangle \equiv \prod_{k\in\mathcal{N}-1/2}\left[\tfrac{1}{2}\left(1+\mathrm{i}c_k^\dagger d_{-k}^\dagger\right)\left(1+\mathrm{i}d_k^\dagger c_{-k}^\dagger\right)\right]|0\rangle. \tag{2.28}$$

Evidently, at small r (far past), the ground state approaches to the bare vacuum $|0\rangle$. In particular, if $r \gg 1$ and $r_0 \ll 1$ equation (2.25) gives

$$\begin{aligned}\alpha_k(r) &\to \left(\frac{2}{r_0}\right)^k \frac{\mathrm{e}^r}{\sqrt{4\pi}}\Gamma\left(k+\tfrac{1}{2}\right)\\ \beta_k(r) &\to \left(\frac{2}{r_0}\right)^k \frac{\mathrm{e}^r}{\sqrt{4\pi}}\Gamma\left(k+\tfrac{1}{2}\right).\end{aligned} \tag{2.29}$$

3. The VEVs of the exponential fields in the free-fermion case

As is shown in [4], The VEV (1.6)

$$G_a = \left\langle \mathrm{e}^{\mathrm{i}a\varphi}(0)\right\rangle = \frac{\int\mathcal{D}\varphi\, \mathrm{e}^{\mathrm{i}a\varphi}\mathrm{e}^{-\mathcal{S}_{SG}(\varphi)}}{\int\mathcal{D}\varphi\, \mathrm{e}^{-\mathcal{S}_{SG}(\varphi)}} \tag{3.1}$$

where $\mathcal{S}_{SG}$ is the action (1.1), can be expressed alternatively in terms of the appropriately regularized (see below) Euclidean functional integral over the Dirac fermions

$$G(a) = \frac{\int_{\mathcal{F}_a} \left[\mathcal{D}\psi\, \mathcal{D}\overline{\psi}\right] \mathrm{e}^{-\mathcal{A}_{MTM}}}{\int_{\mathcal{F}_0} \left[\mathcal{D}\psi\, \mathcal{D}\overline{\psi}\right] \mathrm{e}^{-\mathcal{A}_{MTM}}} \tag{3.2}$$

where $\mathcal{A}_{MTM}$ is the Euclidean action (2.6). The functional integral in the numerator of (3.2) is taken over the space $\mathcal{F}_a$ of those twisted field configurations $\psi\,(z,\bar{z})$ and $\overline{\psi}\,(z,\bar{z})$, which transform as

$$\psi\,(z,\bar{z}) \to \mathrm{e}^{\mathrm{i}2\pi a/\beta}\psi\,(z,\bar{z}) \qquad \overline{\psi}\,(z,\bar{z}) \to \mathrm{e}^{-\mathrm{i}2\pi a/\beta}\overline{\psi}\,(z,\bar{z}) \tag{3.3}$$

when continued analytically around the point $z = 0$ in an anticlockwise direction [4]. The reason for this is the non-trivial monodromy of Dirac fields with respect to the exponential fields $\exp \mathrm{i}a\varphi(0)$. It is easy to see that to impose such twisted boundary conditions on Dirac fields, one has to shift Fourier mode indices as follows:

$$\begin{aligned} &k \to k - \frac{a}{\beta} \quad \text{in sector 1 (i.e. in } c_k, d_k \text{ sector)} \\ &k \to k + \frac{a}{\beta} \quad \text{in sector 2 (i.e. in } d_k, c_k \text{ sector).} \end{aligned} \tag{3.4}$$

For example, the Fourier decomposition of the field $\psi_L(\theta)$ takes the form (cf with the first line of (2.9))

$$\psi_L(\theta) = \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathcal{N}-1/2} \left(c_{k-\alpha}\mathrm{e}^{-\mathrm{i}(k-\alpha)\theta} + d^{\dagger}_{k-\alpha}\mathrm{e}^{\mathrm{i}(k-\alpha)\theta}\right) \tag{3.5}$$

where

$$\alpha \equiv \frac{a}{\beta}. \tag{3.6}$$

With such shifts, all the results of the previous sector remain valid since we have never used the arithmetical properties of the Fourier mode indices.

In the radial Hamiltonian formalism the regularized version of the functional integral (3.2) may be represented as

$$G(a, r_0) = \lim_{r\to\infty} \frac{\langle\infty|\, S(r, r_0)\, |0\rangle_a}{\langle\infty|\, S(r, r_0)\, |0\rangle} \tag{3.7}$$

where the matrix element of the evolution operator $S(r, r_0)$ in the numerator is taken in the twisted sector (this is indicated by the lower index a). To regularize the expression, in (3.7) we have assumed that the evolution begins at some small r_0. A simple conformal field theory consideration†, which takes into account the fact that the conformal dimension of the field $\mathrm{e}^{\mathrm{i}a\varphi}$ is a^2, leads to

$$G_0(a) = \lim_{r_0\to 0} (r_0)^{-2a^2}\, G_0(a, r_0). \tag{3.8}$$

In the general case it is not known how to calculate the functional integral (3.2) or the matrix elements in (3.7) exactly. Below (3.7) is evaluated at the free-fermion point $g = 0$. As in this case we already know the time evolution of every constituent of the vacuum $|0\rangle$ (see

† In the limit $r \to 0$ the action (2.6) describes the massless Thirring model, which is well known to be conformal invariant.

equation (2.20)) from the previous section, it is not difficult to pick up all the necessary factors from (2.29) with appropriate shifts of Fourier mode indices and obtain

$$G_0(a, r_0) = \lim_{r\to\infty} \prod_{k\in\mathcal{N}-1/2} \left(\frac{2}{r_0}\right)^{k+\alpha} \frac{\mathrm{e}^r}{\sqrt{4\pi}} \Gamma\left(k+\alpha+\tfrac{1}{2}\right) \prod_{l\in\mathcal{N}-1/2} \left(\frac{2}{r_0}\right)^{l-\alpha} \frac{\mathrm{e}^r}{\sqrt{4\pi}} \Gamma\left(l-\alpha+\tfrac{1}{2}\right)$$
$$\times\left[\prod_{k\in\mathcal{N}-1/2} \left(\frac{2}{r_0}\right)^{k} \frac{\mathrm{e}^r}{\sqrt{4\pi}} \Gamma\left(k+\tfrac{1}{2}\right) \prod_{l\in\mathcal{N}-1/2} \left(\frac{2}{r_0}\right)^{l} \frac{\mathrm{e}^r}{\sqrt{4\pi}} \Gamma\left(l+\tfrac{1}{2}\right)\right]^{-1} \qquad (3.9)$$

where we have endowed G with the subscript 0 in order to emphasize that the free-fermion case $g = 0$ is considered. We will be careful, when evaluating infinite products in (3.9) and treat the ill-defined sums like $\sum_{i=0}^{\infty}(i \pm a)$ by means of Riemann ζ-function regularization. Let us remind the reader that

$$\zeta\,(z, a) = \sum_{i=0}^{\infty} \frac{1}{(i+a)^z} \qquad (3.10)$$

and

$$\zeta\,(-1, a) + \zeta\,(-1, -a) - 2\zeta\,(-1, 0) = -a^2. \qquad (3.11)$$

To calculate the remaining infinite products of Γ-functions (also divergent, if treated literally), it is convenient to use the integral representation

$$\ln\Gamma(\nu) = \int_0^{\infty} \left[\frac{\mathrm{e}^{-\nu t} - \mathrm{e}^{-t}}{1-\mathrm{e}^{-t}} + (\nu-1)\,\mathrm{e}^{-t}\right] \frac{\mathrm{d}t}{t}. \qquad (3.12)$$

As a result we obtain simple geometric progressions coming from the first term of equation (3.12) and contributions coming from the second term, which can be easily handled by applying ζ-function regularization once more. The final expression has the form

$$G_0(a, r_0) = \left(\frac{r_0}{2}\right)^{\alpha^2} \exp\int_0^{\infty} \left[\frac{\sinh^2(\alpha t)}{\sinh^2 t} - \alpha^2 \mathrm{e}^{-2t}\right] \frac{\mathrm{d}t}{t} \qquad (3.13)$$

or, taking into account equations (3.8) and (3.6) with the free-fermion point value $\beta = 1/\sqrt{2}$,

$$G_0(a) = \left(\frac{M}{2}\right)^{2a^2} \exp\int_0^{\infty} \left[\frac{\sinh^2(\sqrt{2}at)}{\sinh^2 t} - 2a^2 \mathrm{e}^{-2t}\right] \frac{\mathrm{d}t}{t}. \qquad (3.14)$$

This is in full agreement with the result, obtained by Lukyanov and Zamolodchikov in [4], using an angular quantization technique.

4. VEV of the exponential field in the first order of perturbation theory

In this section we calculate the VEV (1.6) in first order of the MTM's coupling constant g. The perturbation is given by the last term of the Hamiltonian (2.7):

$$H_{int} = 2g \int_0^{2\pi} N\left(\Psi_L^+\Psi_L\Psi_R^+\Psi_R\right) \mathrm{d}\theta \qquad (4.1)$$

where we have denoted by $N(\cdots)$ an appropriately regularized product of local fields at a coinciding point. One has to chose such a regularization, which will not break the translational invariance of the theory when transformed back to the initial Euclidean coordinates x^1, x^2. The conventional normal ordering with respect to creation–annihilation operators fails to satisfy this condition because of the non-trivial time dependence of the physical vacuum. Instead, a correctly regularized product one obtains suppressing all the contractions among fields inside the normal ordering symbol $N(\cdots)$

$$\begin{aligned}
N\big(\psi_L^\dagger\psi_L\psi_R^\dagger\psi_R\big) &= \psi_L^\dagger\psi_L\psi_R^\dagger\psi_R - \langle\psi_L^\dagger\psi_L\rangle_0\psi_R^\dagger\psi_R - \langle\psi_R^\dagger\psi_R\rangle_0\psi_L^\dagger\psi_L + \langle\psi_L^\dagger\psi_R\rangle_0\psi_R^\dagger\psi_L \\
&\quad + \langle\psi_R^\dagger\psi_L\rangle_0\psi_L^\dagger\psi_R + \langle\psi_L^\dagger\psi_L\rangle_0\langle\psi_R^\dagger\psi_R\rangle_0 - \langle\psi_L^\dagger\psi_R\rangle_0\langle\psi_R^\dagger\psi_L\rangle_0 \\
&= \,:\psi_L^\dagger\psi_L\psi_R^\dagger\psi_R: + \langle:\psi_L^\dagger\psi_R:\rangle_0 :\psi_R^\dagger\psi_L: + \langle:\psi_R^\dagger\psi_L:\rangle_0 :\psi_L^\dagger\psi_R: \\
&\quad - \langle:\psi_L^\dagger\psi_L:\rangle_0 :\psi_R^\dagger\psi_R: - \langle:\psi_R^\dagger\psi_R:\rangle_0 :\psi_L^\dagger\psi_L: - \langle:\psi_L^\dagger\psi_R:\rangle_0\langle:\psi_R^\dagger\psi_L:\rangle_0 \\
&\quad + \langle:\psi_L^\dagger\psi_L:\rangle_0\langle:\psi_R^\dagger\psi_R:\rangle_0
\end{aligned} \tag{4.2}$$

where the symbol : : denotes the ordinary normal ordering with respect to the creation-annihilation operators c, d, $c^\dagger$, $d^\dagger$, and the expectation value of any operator X is defined by

$$\langle X\rangle_0 \equiv \frac{\langle\infty|S(R,r)XS(r,r_0|0\rangle)}{\langle\infty|S(R,r_0)|0\rangle} \tag{4.3}$$

with all matrix elements taken in the untwisted sector. In (4.3) a small initial time r_0 and a large final time R_0 are introduced in order to keep intermediate expressions finite. R and r_0 eventually should be sent to 0 and ∞, respectively. Let us also emphasize the appearance of an explicit r dependence in (4.3) and, therefore, in (4.2), which reflects the inhomogeneity of 'time' in our scheme of quantization.

The standard time-dependent perturbation theory in first order of the coupling constant g gives

$$G(a,r_0) = \lim_{R\to\infty}\left(\langle\infty|S(R,r_0)|0\rangle_a + \int_{r_0}^{R}\langle\infty|S(R,r)\,H_{int}\,S(r,r_0)\,|0\rangle_a\,\frac{\mathrm{d}r}{r}\right). \tag{4.4}$$

As we have already obtained explicit expressions for time evolution of states from various sectors of Fock space in section 2, it is not difficult to calculate the matrix element under the integral in equation (4.4)

$$\begin{aligned}
G(a,r_0) &= \lim_{R\to\infty}\langle\infty|S(R,r_0)|0\rangle_a \\
&\quad\times\Bigg\{1+\frac{g}{\pi}\sum_{k,l=0}^{\infty}\int_{r_0}^{R} r\,\mathrm{d}r\big[2I_{k+1-a}K_{k-a}I_{l+1+a}K_{l+a} - I_{k+1-a}K_{k-a}I_{l+1-a}K_{l-a} \\
&\quad - I_{k+1+a}K_{k+a}I_{l+1+a}K_{l+a} - I_{k-a}K_{k-a}I_{l-a+1}K_{l-a+1} - I_{k+a}K_{k+a}I_{l+a+1}K_{l+a+1} \\
&\quad - I_{k+1-a}K_{k+1-a}I_{l+1+a}K_{l+1+a} - I_{k+a}K_{k+a}I_{l-a}K_{l-a} + I_{k+1}K_{k+1}I_{l-a+1}K_{l-a+1} \\
&\quad + I_kK_kI_{l-a+1}K_{l-a+1} + I_kK_kI_{l+a}K_{l+a} + I_kK_kI_{l+a+1}K_{l+a+1} + I_kK_kI_{l-a}K_{l-a} \\
&\quad + I_{k+1}K_{k+1}I_{l+a+1}K_{l+a+1} + I_{k+1}K_{k+1}I_{l-a}K_{l-a} + I_{k+1}K_{k+1}I_{l+a}K_{l+a} \\
&\quad - I_{k+1}K_{k+1}I_{l+1}K_{l+1} - I_{k+1}K_{k+1}I_lK_l - I_kK_kI_{l+1}K_{l+1} - I_kK_kI_lK_l\big]\Bigg\}
\end{aligned} \tag{4.5}$$

where the prefactor $\langle\infty|S(R,r_0)|0\rangle_a$ is given by equation (3.13) (in (3.13) we have to insert $\beta = \frac{1}{\sqrt{2}}(1-g/2\pi+\mathrm{o}(g))$ and expand the resulting expression over g up to linear term). The calculation of integrals from the quartic products of modified Bessel functions is presented in the appendix. Using its results we obtain

$$G(a, r_0) = \left(\tfrac{1}{2}r_0\right)^{\alpha^2} \exp\left\{\int_0^\infty \left(\frac{\sinh^2(\alpha t)}{\sinh^2 t} - \alpha^2 e^{-2t}\right)\frac{dt}{t}\right\}$$
$$\times\left\{1 + \frac{g}{2\pi}\left[2\alpha^2 \log\frac{r_0}{2} + \int_0^\infty \left(\frac{\alpha\sinh(2\alpha t)}{2\sinh^2 t} - \frac{\alpha^2}{t}e^{-2t}\right) dt\right] + O(g^2)\right\}$$
$$\times\left\{1 + \frac{g}{\pi}\left[\int_0^\infty \sum_{k,l=0}^{\infty}\left(\frac{8\cosh^2 t\sinh^2\alpha t - 4\sinh^2 2\alpha t}{\sinh 2t}e^{-2(k+l+1)t}\right) dt\right]\right.$$
$$\left. + O(g^2)\right\}. \tag{4.6}$$

Now performing a summation over k and l in the third line of equation (4.6) we obtain an integral which diverges logarithmically at $t = 0$. In fact, the same problem we have faced when carrying out calculations at the free-fermion point. Indeed, the product in (3.9) diverges for large k, l, but we have overcome this difficulty using ζ-function regularization inside the integral representation of the Γ-function (3.12). Here there is no necessity to carry out a similar regularization. Indeed, noticing that various regularization schemes could differ from each other at most by a term $\sim a^2$, and that the coefficient of $-a^2/2$ in the expansion of $\langle e^{ia\phi}\rangle$ is just the VEV $\langle\phi^2\rangle$, already calculated in [4] using a standard Feynman diagram technique with the result (below $\gamma = 0.577\,216\ldots$ is the Euler constant)

$$\langle\phi^2(0)\rangle = -4(1 + \gamma + \log(M/2)) + \frac{g}{\pi}(7\zeta(3) - 2) + O(g^2) \tag{4.7}$$

we can simply cut the above-mentioned integral over t on the lower bound by a small cut-off and require that the undefined coefficient of $-a^2/2$ take the value predicted by equation (4.7). The final result has the form

$$\left\langle e^{ia\phi(0)}\right\rangle = \left(\tfrac{1}{2}M\right)^{\alpha^2} \exp\left\{\int_0^\infty \left(\frac{\sinh^2(\alpha t)}{\sinh^2 t} - \alpha^2 e^{-2t}\right)\frac{dt}{t}\right\}$$
$$\times\left\{1 + \frac{g}{\pi}\left[\int_0^\infty \left(\frac{\alpha\sinh(2\alpha t)}{2\sinh^2 t} - \frac{\sinh^2(\alpha t)}{\sinh^3 t}\right) dt - 2\alpha^2\log 2\right] + O(g^2)\right\} \tag{4.8}$$

which is in complete agreement with the Lukyanov–Zamolodchikov formula (1.8).

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Appendix

It appears that the Hankel transforms are appropriate tools allowing us to perform the integration over r in (4.5). Roughly speaking, in polar coordinates the Hankel transforms play the same role as the ordinary Fourier transforms in the Cartesian one.

Let us briefly recall the main formulae concerning the Hankel transforms (for details see [13] and references therein). The νth order ($\nu > -1$) direct and inverse Hankel transforms of the function $f(x)$ defined on $(0, \infty)$ are given by

$$f(x) = \int_0^\infty J_\nu(sx)\widetilde{f}_\nu(s)s\,\mathrm{d}s \tag{A.1}$$

$$\widetilde{f}_\nu(s) = \int_0^\infty J_\nu(sx)f(x)x\,\mathrm{d}x \tag{A.2}$$

where J_ν is the Bessel function. In complete analogy with the case of the Fourier transform, it follows from (A.1) and (A.2), that the 'scalar product' of any two functions $f(x)$, $g(x)$ coincides with that of their images (since the functions we are dealing with are regular in the interval $(0, \infty)$, the only thing one has to care about is the convergence of integrals at the extreme points 0 and ∞).

$$\int_0^\infty f(x)g(x)x\,\mathrm{d}x = \int_0^\infty \widetilde{f}_\nu(s)\widetilde{g}_\nu(s)s\,\mathrm{d}s. \tag{A.3}$$

To use (A.3) for the calculation of the integral in equation (4.5) we need to know Hankel images of the functions $I_\nu(x)K_\nu(x)$ and $I_{\nu+1}(x)K_\nu(x)$ which can be easily obtained from the formula [13]

$$K_{-\nu}(x)I_\mu(x) = \int_0^\infty J_{-\nu+\mu}\,(2x\sinh t)\,\mathrm{e}^{-(\nu+\mu)t}\,\mathrm{d}t - \mathrm{Re}(\nu+\mu) < \tfrac{3}{2} \qquad \mathrm{Re}(-\nu+\mu) > -1 \tag{A.4}$$

namely

$$I_l(x)K_l(x) = \int_0^\infty J_0(xs)\frac{1}{s\sqrt{s^2+4}}\mathrm{e}^{-2lt(s)}s\,\mathrm{d}s \tag{A.5}$$

$$I_{l+1}(x)K_l(x) = \int_0^\infty J_1(xs)\frac{1}{s\sqrt{s^2+4}}\mathrm{e}^{-(2l+1)t(s)}s\,\mathrm{d}s \tag{A.6}$$

where $t(s)$ is defined by

$$2\sinh t = s \qquad \mathrm{d}t = \frac{\mathrm{d}s}{\sqrt{s^2+4}}. \tag{A.7}$$

Though the direct application of equation (A.3) to each term of (4.5) at first sight seems to be problematic due to the logarithmic divergence of the integral at large r, it nevertheless leads to a correct result, because of mutual cancellation of these divergences by various terms.

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